Quasinormal modes and entropy spectrum of three dimensional Gödel black hole

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Abstract

We have studied perturbations of scalar and spinor field in the background of three dimensional Gödel black hole. The wave equations are shown to be exactly solvable in terms of hypergeometric functions. The quasinormal modes are analytically calculated by imposing the Dirichlet boundary condition at spatial infinity, which are shown to be of the same form in both cases. By considering the physical interpretation of quasinormal modes, we obtain the consistent transition frequencies from the quasinormal modes of scalar and spinor field. As an application of quasinormal modes, we have also investigated the area and entropy quantization of three dimensional Gödel black hole. By choosing the conserved mass of Gödel black hole properly, the entropy spectrum are shown to be equally-spaced.

PACS numbers:

Keywords: Gödel black hole, quasinormal modes, entropy quantization

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I. INTRODUCTION

The studying of matter field perturbation in black hole background can provide deep insight into quantum properties of black hole. By using the techniques of quantum field theory in curved spacetime, Hawking [1] made the intriguing observation that black hole can radiate thermally like a blackbody at the temperature propotional to the surface gravity of the event horizon. This discovery confirms the Bekenstein's conjecture [2] that black hole have a thermodynamics entropy propotional to the area of horizon.

Quasinormal mode is an interesting topic in black hole perturbation theory. It can be regarded as the resonance of a black hole under a small external field perturbation. Quasinormal modes can be computed by solving the perturbation field equation in a fixed background spacetime. Quasinormal modes are damped modes with complex frequencies, which depend only on the parameters of black hole. The studying of quasinormal modes has a long history, and there has been extensive work done to compute quasinormal modes in various black hole backgrounds. For a recent review, one can refer to [3, 4].

Inspired by the AdS/CFT correspondence [5–7], quasinormal modes of asymptotically AdS black holes are related to the retarded Green's functions of the dual thermal conformal field theories. It was established [8] that the relaxation time of a thermal state of the boundary thermal field theory is proportional to the inverse of the imaginary part of quasinormal modes of the dual gravity background. Besides their theoretical importance, quasinormal modes also have their astronomical applications. One can get a deeper understanding of the black holes in nature by detecting signals of quasinormal modes using the gravitational wave detectors [9].

In recent years, it was proposed that quasinormal modes can also be used to quantize the area and the entropy of a black hole. Bekenstein [10] is the first to propose that the area of a black hole should be quantized. He showed that the black hole horizon area is an adiabatic invariant, which leads to a discrete spectrum under a suitable quantization process. Hod's proposal of the relationship between quasinormal modes and area quantization of a black hole suggests that the real part of the asymptotic quasinormal modes can be regarded as a transition frequency in the semiclassical limit [11]. Latter, Kunstatter [12] suggested that the horizon area of a black hole can be quantized by using an adiabatic invariant I. For a system with energy E and vibrational frequency $\omega(E)$, the adiabatic invariant $I = \int dE/\omega(E)$ is

quantized by identifying the real part of the quasinormal modes as the transition frequency and using Bohr-Sommerfeld quantization condition. More recently, Maggiore [13] proposed that, for a highly damped modes, the imaginary part of quasinormal modes should be taken as the transition frequency. Because of these developments, there have been many works to compute quasinormal modes and area spectra in various types of black holes [14].

Three dimensional Gödel spacetime [15] is an exact solution to Einstein-Maxwell theory in 2+1 dimensions with a negative cosmological constant and a Chern-Simons term. This theory can be viewed as a lower dimensional toy model for the bosonic part of five dimensional supergravity theory, since the field content and the couplings of both theories are similar. Three dimensional Gödel black holes display the same peculiar properties as their higher dimensional counterparts[15]. The rotating black hole solutions on the Gödel background in the context of five-dimensional supergravity theory have attracted a lot of attention [16]. More recently, the quasinormal modes and stability of five-dimensional rotating Gödel black holes are investigated by R. A. Konoplya et.al. in [17]. In this paper, we will study the perturbations of scalar field and spinor field in the background of three dimensional Gödel black hole.

Most calculations of quasinormal modes are numerical due to the difficulties in solving the differential equations. However, in three dimensions, there have been several papers where quasinormal modes are computed analytically. The well-known BTZ black hole has been studied with exact results [18]. Recent studies show that the quasinormal modes of warped AdS black holes can also be analytically calculated [19, 20]. We will show that the wave equations of scalar field and spinor field in three dimensional Gödel black hole can also be exactly solved in terms of hypergeometric functions. The quasinormal modes of scalar field and spinor field are analytically calculated by imposing the Dirichlet boundary condition at the spatial infinity, which are shown to be of the same form in both cases. By considering the physical interpretation of quasinormal modes, we obtain the consistent transition frequencies from the quasinormal modes of scalar field and spinor field. As an application of quasinormal modes, we use the proposal of area quantization of rotating black hole in [21, 22] to investigate area and entropy quantization of three dimensional Gödel black hole. It is shown that, when the conserved mass of three dimensional Gödel black hole is taken as the parameter ν, the area and entropy can be quantized and the spectra are equally-spaced.

This paper is arranged as following. In Sec. II, we give a brief review of three dimensional Gödel black hole. In Sec. III and IV, we consider the perturbations of scalar field and spinor field in the background of three dimensional Gödel black hole and calculate the corresponding quasinormal modes analytically by solving the equation of motion. In Sec. V, using the computed quasinormal modes, area and entropy spectrum of three dimensional Gödel black hole are presented. The last section is devoted to summary and conclusion.

II. THREE DIMENSIONAL GÖDEL BLACK HOLE

In this section, we will firstly give a brief review of the geometric and thermodynamic properties of three dimensional Gödel black hole. Three dimensional Gödel spacetime is an exact solution to Einstein-Maxwell theory in 2+1 dimensions with a negative cosmological constant and a Chern-Simons term. The action is given by

$$I = \frac{1}{16\pi G} \int d^3x \left[\sqrt{-g} \left(R + \frac{2}{l^2} - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right) - \frac{\alpha}{2} \epsilon^{\mu\nu\rho} A_\mu F_{\nu\rho} \right] , \tag{1}$$

where G is the three dimensional gravitational constant, the parameter l is related to the cosmological constant Λ by $\Lambda = -1/l^2$ and α is the Chern-Simons coupling constant.

It has been shown that the equations of motion derived from this action admit the black hole solution [15], where the metric and the non-vanishing component of gauge potential A_μ are given by

$$\begin{aligned} ds^2 &= dt^2 - 4\alpha r dt d\varphi + \left[8G\nu r - (1-\alpha^2 l^2)\frac{2r^2}{l^2} - \frac{4GJ}{\alpha} \right] d\varphi^2 \\ &\quad + \left[(1+\alpha^2 l^2)\frac{2r^2}{l^2} - 8G\nu r + \frac{4GJ}{\alpha} \right]^{-1} dr^2 , \end{aligned} \tag{2}$$

$$A_\varphi = -\frac{4GQ}{\alpha} + \sqrt{1-\alpha^2 l^2}\frac{2r}{l} . \tag{3}$$

The parameters ν, J and Q are integral constants, which may be related to the mass, angular momentum and charge of black hole.

Note that because of the presence of a nontrivial gauge field, the asymptotic geometry of three dimensional Gödel black hole does not behave as either de Sitter or anti-de Sitter. It is also obvious that it is not asymptotic to the warped AdS spacetime. The asymptotic symmetry algebra of this spacetime has been studied in [23], which turns out to be the semi-direct sum of the diffeomorphisms on the circle with two loop algebras. The covariant Poisson

bracket of the conserved charges associated with the generators of asymptotic symmetry group is shown to be centrally extended to the semi-direct sum of a Virasoro algebra and two affine algebras. The aspect of holographic dual between this black hole and conformal field theory should be further studied, which is not considered in the present paper.

The black hole has two horizons, i.e. the inner and the outer event horizons $r_\pm$, which are determined by the equation

$$(1+\alpha^2 l^2)\frac{2r^2}{l^2} - 8G\nu r + \frac{4GJ}{\alpha} = 0 \ . \tag{4}$$

The solutions give the locations of event horizons

$$r_\pm = \frac{l^2}{1+\alpha^2 l^2}\left[2G\nu \pm \sqrt{4G^2\nu^2 - \frac{2GJ}{\alpha}\frac{(1+\alpha^2 l^2)}{l^2}}\right] \ . \tag{5}$$

The outer and the inner event horizons are the coordinate singularities of the metric, which can be eliminated by a proper coordinates transformation.

Now, Let us discuss the thermodynamics of the black hole. The Hawking temperature T_H is computed as

$$T_H \ = \frac{(1+\alpha^2 l^2)}{4\pi\alpha l^2}\frac{(r_+ - r_-)}{r_+} \ . \tag{6}$$

The angular velocity at the outer event horizon of the black hole is given by

$$\Omega_H = \frac{1}{2\alpha r_+} \ . \tag{7}$$

The entropy of the black hole can be calculated by using the Wald's formalism, which is the quarter of area of the outer horizon for Einstein gravity

$$S = \frac{A}{4G} = \frac{\pi\alpha r_+}{G} \ . \tag{8}$$

Here, we present an intuitive derivation of conserved charges of the black hole. By differentiating the expression of the outer event horizon r_+, one can deduce the following relationship

$$d\nu = TdS + \Omega_H dJ \ . \tag{9}$$

If one identifies the parameter ν as the mass M and the parameter J as the angular momentum of the black hole, this differential relationship can be treated as the first law of black

hole thermodynamics. It has been shown in [15] that, via the rigorous definition of conserved charges and tensor calculation, the parameter ν is the conserved quantity associated to the killing vector ∂_t. Here, we just present a simple and intuitive derivation of black hole mass. However, it is observed in [23] that, under the change of coordinates $r \to -r$, $\phi \to -\phi$, the solutions with the parameters (ν, J, Q) can be changed to the solutions with the parameters $(-\nu, J, -Q)$. So the conserved quantity ν does not provide a satisfactory definition of black hole mass. They found the definition of black hole mass as $\mu = 2G\nu^2$ can properly recover this shortcoming. As we will done in the Sec.V, the conserved quantity ν has been taken as the energy of black hole in the process of entropy quantization. The result shows this identification is effective and reasonable, at least, in handling with the quantization of black hole entropy.

One should also note that the electrostatic potential and charge does not appear in the differential form of thermodynamics first law. In fact, one can impose a background electrostatic potential to make the total electrostatic potential vanish. Especially, the electric parameter Q is an integral constant and dose not appear in the metric function. So, when calculating the quasinormal modes of perturbation fields, we will not consider the coupling between the gauge field A_μ and the matter fields.

III. QUASINORMAL MODES OF SCALAR FIELD

In this section, we will analytically calculate quasinormal modes of scalar field perturbation in the background of three dimensional Gödel black hole. We consider the equation of motion for scalar field perturbation, which is given by the Klein-Gordon equation

$$\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\partial_\nu\Phi\right) + \mu^2\Phi = 0, \tag{10}$$

with μ being the mass of scalar field.

According to the symmetry of background spacetime, i.e. the existence of two killing vectors ∂_t and ∂_φ, we can expand the scalar field $\Phi(t, r, \varphi)$ as following

$$\Phi(t, r, \varphi) = e^{-i\omega t + im\varphi}R(r)\ . \tag{11}$$

Then, by substituting this expression for the scalar field perturbation into Eq.(10), one can

get the radial wave equation after some algebra

$$\Delta\frac{d}{dr}\left(\Delta\frac{d}{dr}R(r)\right)+\left(\omega^2(4\alpha^2r^2-\Delta)-4\omega m\alpha r+m^2-\mu^2\Delta\right)R(r)=0, \tag{12}$$

where, for latter convenience, we have introduced the function $\Delta(r)=\lambda(r-r_+)(r-r_-)$ with $\lambda=2(1+\alpha^2l^2)/l^2$. This equation can be analytically solved by the hypergeometric function, which provides us with an exact calculation of quasinormal modes of scalar field perturbation.

In order to solve the radial wave equation, it is convenient to introduce the variable z as

$$z=\frac{r-r_+}{r-r_-}\ . \tag{13}$$

Then, the radial wave equation can be rewritten in the form of hypergeometric equation

$$z(1-z)\frac{d^2R(z)}{dz^2}+(1-z)\frac{dR(z)}{dz}+\left(\frac{A}{z}+B+\frac{C}{1-z}\right)R(z)=0\ , \tag{14}$$

where the parameters A, B and C are given by

$$\begin{aligned}
A &= \frac{(2\alpha r_+\omega-m)^2}{\lambda^2(r_+-r_-)^2}\ ,\\
B &= -\frac{(2\alpha r_-\omega-m)^2}{\lambda^2(r_+-r_-)^2}\ ,\\
C &= \frac{4\alpha^2\omega^2}{\lambda^2}-\frac{\omega^2+\mu^2}{\lambda}\ .
\end{aligned} \tag{15}$$

In terms of the definition of quasinormal modes, the solution of perturbation field must be purely ingoing near the horizon of black hole. So we are just interested in the solution with the ingoing boundary condition at the horizon. The solution of radial wave equation with the ingoing boundary condition is given explicitly by the hypergeometric function

$$R(z)=z^{\alpha_s}(1-z)^{\beta_s}F(a_s,b_s,c_s,z)\ , \tag{16}$$

where

$$\alpha_s=-i\sqrt{A}\ ,\quad \beta_s=\frac{1}{2}-\sqrt{\frac{1}{4}-C}\ , \tag{17}$$

and

$$\begin{aligned}
c_s &= 2\alpha_s+1\ ,\\
a_s &= \alpha_s+\beta_s+i\sqrt{-B}\ ,\\
b_s &= \alpha_s+\beta_s-i\sqrt{-B}\ .
\end{aligned} \tag{18}$$

So, we have shown that the equation of motion for scalar field perturbation in the background of three dimensional Gödel black hole can be exactly solved in terms of hypergeometric function after the partial wave decomposition. Now, we will analyse the asymptotic properties of the solution at spatial infinity and calculate the corresponding quasinormal modes by imposing the Dirichlet boundary condition at spatial infinity.

By using the following transformation property of hypergeometric function

$$\begin{aligned} F(a,b,c;z) &= \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}F(a,b,a+b-c+1;1-z) \\ &+(1-z)^{c-a-b}\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}F(c-a,c-b,c-a-b+1;1-z)\ , \end{aligned} \tag{19}$$

one can find the leading asymptotic behaviour of the solution $R(z)$ at the spatial infinity (i.e., $z \to 1$)

$$R(z) \simeq z^{\alpha_s}(1-z)^{\beta_s}\frac{\Gamma(c_s)\Gamma(c_s-a_s-b_s)}{\Gamma(c_s-a_s)\Gamma(c_s-b_s)}\ . \tag{20}$$

Next, in order to find the quasinormal modes, one has to impose the boundary condition at the asymptotic infinity. The condition that the flux vanishes at asymptotic infinity is just a perfect one. Here, we will use the equivalent Dirichlet condition that the scalar field is vanishing at asymptotic infinity to obtain the quasinormal modes. Imposing the vanishing Dirichlet boundary condition at spatial infinity leads to the following relation

$$c_s - a_s = -n\ , \quad \text{or} \quad c_s - b_s = -n\ , \tag{21}$$

which give the following two equations of quasinormal modes for scalar field perturbation

$$\begin{aligned} &\omega - \frac{m}{\alpha(r_+ + r_-)} + i\frac{\lambda(r_+ - r_-)}{2\alpha(r_+ + r_-)}(n+h_s) = 0, \\ &\omega + i\frac{\lambda}{2\alpha}(n+h_s) = 0, \end{aligned} \tag{22}$$

with

$$h_s = \frac{1}{2} + \sqrt{\frac{\omega^2}{\lambda^2}\left(\lambda - 4\alpha^2\right) + \frac{\mu^2}{\lambda} + \frac{1}{4}}. \tag{23}$$

Quasinormal modes can be regarded as the resonances of a black hole under a small perturbation. They can be obtained by solving the perturbation equation in a fixed background geometry under the proper boundary conditions, which generally results in a complex damping frequency. The second equation of (22) can be easily solved, which gives the explicit

expression for one family of quasinormal mode as

$$\omega = \sqrt{(\lambda - 4\alpha^2)\left(n + \frac{1}{2}\right)^2 - \mu^2 - \frac{\lambda}{4}} - 2i\alpha\left(n + \frac{1}{2}\right) . \tag{24}$$

The another family of quasinormal modes is determined by the first equation. The solution to this equation is tedious, which will not be presented here. From Eq.(24) and the first equation of (22), one can conclude that the quasinormal modes are determined only by the parameters of black hole.

It is noted that Eq.(22) is rather complicated to analyze. In order to analyze the asymptotic properties of quasinormal modes, we adopt a kind of identification from the calculation of quasinormal modes for the warped AdS_3 black hole by B. Chen and Z.-B. Xu in [19]. If making the following identification

$$t \to -\varphi \ , \quad \varphi \to \frac{t}{\alpha(r_+ + r_-)} \ , \tag{25}$$

one can find that the quantum numbers in these two backgrounds satisfy the following relations

$$\tilde{\omega} = -\frac{m}{\alpha(r_+ + r_-)} \ , \quad \tilde{m} = \omega \ . \tag{26}$$

Thus, the equation (22) can give the following two sectors of quasinormal modes

$$\begin{aligned} \tilde{\omega}_L^s &= -\tilde{m} - i\frac{\lambda(r_+ - r_-)}{2\alpha(r_+ + r_-)}(n + \tilde{h}_s) \ , \\ \tilde{\omega}_R^s &= -i\frac{\lambda}{2\alpha}(n + \tilde{h}_s), \end{aligned} \tag{27}$$

with

$$\tilde{h}_s = \frac{1}{2} + \sqrt{\frac{\tilde{m}^2}{\lambda^2}(\lambda - 4\alpha^2) + \frac{\mu^2}{\lambda} + \frac{1}{4}}. \tag{28}$$

The expressions of left sector and right sectors of quasinormal modes will be useful for area and entropy quantization of this black hole. Strictly speaking, the transition frequencies are changed by the identification. But we cannot find any other effective approach to obtain the transition frequency. The transition frequencies obtained by the identification turn out to be effective in the process of entropy quantization. We have successfully quantized the entropy and obtained an equally spaced spectrum which is generally believed to be an exact result.

By analog to the quasinormal modes of BTZ black hole and warped AdS black hole, one can define the left and the right temperatures as

$$T_L = \frac{\lambda(r_+ - r_-)}{4\pi\alpha(r_+ + r_-)} \ ,$$
$$T_R = \frac{\lambda}{4\pi\alpha} \ . \tag{29}$$

The relation of the left and the right temperatures and the Hawking temperature is then given by

$$\frac{1}{T_H} = \frac{1}{T_L} + \frac{1}{T_R} \ . \tag{30}$$

In the context of AdS/CFT corresponding, the dual conformal field theory on the boundary can be separated into two independent left-moving and right-moving sectors at thermal equilibrium with different temperatures. The perturbation fields in the black hole background are dual to the operators in the boundary conformal field theory. The relaxation time τ for a thermal state back to thermal equilibrium in the boundary conformal field theory is related to the imaginary part of quasinormal modes. The dual aspect of black hole and conformal field theory is very interesting for further study, which however is beyond the scope of the present work.

In summary, we have computed the quasinormal modes of scalar field perturbation in this section. In the next section, we will perturb the black hole by a spinor field to calculate its corresponding quasinormal modes.

IV. QUASINORMAL MODES OF SPINOR FIELD

In this section, we will calculate the quasinormal modes of fermionic field perturbation in the background of three dimensional Gödel black hole. For this purpose, we should consider the spinor field Ψ with mass μ in this black hole, which obeys the covariant Dirac equation

$$\gamma^a e_a^\mu \left(\partial_\mu + \frac{1}{2}\omega_\mu^{ab}\Sigma_{ab}\right)\Psi + \mu\Psi = 0 \ , \tag{31}$$

where ω_μ^{ab} is the spin connection, which can be given in terms of the tetrad e_a^μ, $\Sigma_{ab} = \frac{1}{4}[\gamma_a, \gamma_b]$, and $\gamma^0 = i\sigma^2$, $\gamma^1 = \sigma^1$, $\gamma^2 = \sigma^3$, where the matrices σ^k are the Pauli matrices.

In order to analytically solve the covariant Dirac equation, we should firstly choose a proper tetrad field and calculate the corresponding spin connection. According to the metric

of three dimensional Gödel black hole, the tetrad field can be selected to be

$$
\begin{aligned}
e^0 &= \sqrt{\Delta} d\varphi \ , \\
e^1 &= \frac{1}{\sqrt{\Delta}} dr \ , \\
e^2 &= dt - 2\alpha r d\varphi \ .
\end{aligned} \tag{32}
$$

This selection for the tetrad field is not unique, but simple and convenient in the following calculations.

By employing the Cartan structure equation $de^a + \omega^a{}_b \wedge e^b = 0$, one can calculate the spin connection directly. The nonvanishing components of the spin connection are listed as follows

$$
\begin{aligned}
\omega_t^{01} &= \alpha \ , \\
\omega_r^{02} &= \frac{\alpha}{\sqrt{\Delta}} \ , \\
\omega_\varphi^{01} &= \frac{\Delta'}{2} - 2\alpha^2 r \ , \\
\omega_\varphi^{12} &= \alpha\sqrt{\Delta} \ .
\end{aligned} \tag{33}
$$

The inverse of the tetrad field is also needed

$$
\begin{aligned}
e_0 &= \frac{2\alpha r}{\sqrt{\Delta}} \partial_t + \frac{1}{\Delta} \partial_\varphi \ , \\
e_1 &= \sqrt{\Delta} \partial_r \ , \\
e_2 &= \partial_t \ .
\end{aligned} \tag{34}
$$

Assuming that the spinor field takes the form $\Psi = (\psi_+(r), \psi_-(r)) e^{-i\omega t + im\varphi}$ and changing the variables to z, one can finally derive the following equations of motion after some algebra

$$
\begin{aligned}
& z^{\frac{1}{2}}(1-z)\frac{d\psi_+}{dz} + \left[\left(\frac{2i\alpha r_+\omega - im}{\lambda(r_+ - r_-)} + \frac{1}{4}\right) z^{-\frac{1}{2}} + \left(-\frac{2i\alpha r_-\omega - im}{\lambda(r_+ - r_-)} + \frac{1}{4}\right) z^{\frac{1}{2}}\right]\psi_+ \\
& \qquad + \frac{1}{\sqrt{\lambda}}\left(i\omega + \frac{\alpha}{2} + \mu\right)\psi_- = 0 \ , \\
& z^{\frac{1}{2}}(1-z)\frac{d\psi_-}{dz} + \left[\left(-\frac{2i\alpha r_+\omega - im}{\lambda(r_+ - r_-)} + \frac{1}{4}\right) z^{-\frac{1}{2}} + \left(\frac{2i\alpha r_-\omega - im}{\lambda(r_+ - r_-)} + \frac{1}{4}\right) z^{\frac{1}{2}}\right]\psi_- \\
& \qquad + \frac{1}{\sqrt{\lambda}}\left(-i\omega + \frac{\alpha}{2} + \mu\right)\psi_+ = 0 \ .
\end{aligned} \tag{35}
$$

The above equation can also be solved by the hypergeometric function. The solution of $\Psi_+(z)$ with the ingoing boundary condition at the horizon can be explicitly expressed as

$$
\psi_+(z) = z^{\alpha_f}(1-z)^{\beta_f} F(a_f, b_f, c_f, z) \ , \tag{36}
$$

with the parameters as

$$
\begin{aligned}
\alpha_f &= -\frac{i\,(2\alpha r_+\omega - m)}{\lambda(r_+ - r_-)} - \frac{1}{4}\ ,\\
\beta_f &= \frac{1}{2} - \sqrt{\frac{\omega^2}{\lambda^2}\,(\lambda - 4\alpha^2) + \frac{1}{\lambda}\left(\mu + \frac{\alpha}{2}\right)^2}\ ,\\
\gamma_f &= \frac{i\,(2\alpha r_-\omega - m)}{\lambda(r_+ - r_-)} - \frac{1}{4}\ ,\\
a_f &= \alpha_f + \beta_f + \gamma_f\ ,\\
b_f &= \alpha_f + \beta_f - \gamma_f\ ,\\
c_f &= 2\alpha_f + 1\ .
\end{aligned}
\tag{37}
$$

Then, the solution of $\psi_-(z)$ can be obtained by integrating the second equation of (35) as

$$
\begin{aligned}
\psi_-(z) &= z^{-\alpha_f - \frac{1}{2}}(1-z)^{\alpha_f+\gamma_f+1}\int z^{c_f-1}(1-z)^{b_f-c_f-1}F(a_f, b_f, c_f, z)dz\\
&= z^{\alpha_f+\frac{1}{2}}(1-z)^{\beta_f}F(a_f+1, b_f, c_f+1, z)\ .
\end{aligned}
\tag{38}
$$

One can also find the leading asymptotic behaviour ($z \to 1$) of this solution

$$
\begin{aligned}
\psi_+ &\simeq z^{\alpha_f}(1-z)^{\beta_f}\frac{\Gamma(c_f)\Gamma(c_f - a_f - b_f)}{\Gamma(c_f - a_f)\Gamma(c_f - b_f)}\ ,\\
\psi_- &\simeq z^{\alpha_f+\frac{1}{2}}(1-z)^{\beta_f}\frac{\Gamma(c_f+1)\Gamma(c_f - a_f - b_f)}{\Gamma(c_f - a_f)\Gamma(c_f - b_f)}\ .
\end{aligned}
\tag{39}
$$

By imposing the vanishing Dirichlet boundary condition at infinity, one can find the following relation

$$
c_f - a_f = -n\ , \quad c_f - b_f = -n\ ,
\tag{40}
$$

which gives two equations of quasinormal modes for spinor field perturbation

$$
\begin{aligned}
&\omega - \frac{m}{\alpha(r_+ + r_-)} + i\frac{\lambda(r_+ - r_-)}{2\alpha(r_+ + r_-)}(n + h_f) = 0,\\
&\omega + i\frac{\lambda}{2\alpha}(n + h_f - \frac{1}{2}) = 0,
\end{aligned}
\tag{41}
$$

with

$$
h_f = \frac{1}{2} + \sqrt{\frac{\omega^2}{\lambda}\left(1 - \frac{4\alpha^2}{\lambda}\right) + \frac{1}{\lambda}\left(\mu + \frac{\alpha}{2}\right)^2}.
\tag{42}
$$

Similar to the case of scalar field, the first equation is hard to solve. This family of quasinormal modes will not be explicitly given. The second equation can be easily solved,

and the quasinormal modes are given by

$$\omega = \sqrt{(\lambda - 4\alpha^2)n^2 - \left(\mu + \frac{\alpha}{2}\right)^2} - 2i\alpha n \ . \tag{43}$$

One can also make the identification of (25) and (26) to obtain two sectors of quasinormal modes for spinor field similarly to scalar field

$$\begin{aligned} \tilde{\omega}_L^f &= -\tilde{m} - i\frac{\lambda(r_+ - r_-)}{2\alpha(r_+ + r_-)}(n + \tilde{h}_f), \\ \tilde{\omega}_R^f &= -i\frac{\lambda}{2\alpha}(n + \tilde{h}_f - \frac{1}{2}), \end{aligned} \tag{44}$$

with

$$\tilde{h}_f = \frac{1}{2} + \sqrt{\frac{\tilde{m}^2}{\lambda}\left(1 - \frac{4\alpha^2}{\lambda}\right) + \frac{1}{\lambda}\left(\mu + \frac{\alpha}{2}\right)^2}. \tag{45}$$

One can see that the quasinormal modes of spinor field exhibit the same asymptotic properties as that of scalar field when n is very large. In the next section, we will use this observation to quantize the area and the entropy of three dimensional Gödel black hole.

V. AREA AND ENTROPY QUANTIZATION USING QUASINORMAL MODES

The two families of the quasinormal modes (27) and (44) of the three dimensional Gödel black hole for scalar field and spinor field at large n for a fixed $|m|$, in particular for $n \gg |m|$, can give the asymptotic properties as

$$\begin{aligned} \tilde{\omega}_L &= -i\frac{\lambda(r_+ - r_-)}{2\alpha(r_+ + r_-)}n, \\ \tilde{\omega}_R &= -i\frac{\lambda}{2\alpha}n. \end{aligned} \tag{46}$$

This asymptotic properties of the two families of quasinormal modes gives two possible transition frequencies, ω_{Lc} and ω_{Rc}. We find two transition frequencies corresponding to each quasinormal mode

$$\begin{aligned} \omega_{Lc} &= \frac{\lambda(r_+ - r_-)}{2\alpha(r_+ + r_-)}, \\ \omega_{Rc} &= \frac{\lambda}{2\alpha}. \end{aligned} \tag{47}$$

Based on Bohr's correspondence principle, it is proposed in [21, 22] that the transition frequency ω_c of a black hole in the semiclassical limit can be considered as the oscillation

frequency in a classical system of periodic motion. Then, the action variable of the corresponding classical system with energy E and transition frequency ω_c is identified and quantized via Bohr-Sommerfeld quantization in the semiclassical limit as follows

$$\mathfrak{I} = \int \frac{dE}{\omega_c} = n\hbar \, , \quad (n = 0, 1, 2, \cdots) \, . \tag{48}$$

Because the change of the energy E of the system is the change of the mass M of the black hole, one can finally obtain, in the semiclassical limit, the quantization condition for a black hole

$$\mathfrak{I} = \int \frac{dM}{\omega_c} = n\hbar \, , \quad (n = 0, 1, 2, \cdots) \, . \tag{49}$$

This formula holds for a black hole with quasinormal modes regardless of whether it is rotating or not. We will use this quantization condition to obtain the discrete area and entropy spectra of the three dimensional Gödel black hole.

In the present case, we should consider the two action variables corresponding to each possible transition frequency. The two quantization conditions are given by

$$\begin{aligned}
\mathfrak{I}_L &= \int \frac{dM}{\omega_{Lc}} \\
&= \frac{\alpha}{G\lambda}\sqrt{4G^2\nu^2 - \frac{2GJ}{\alpha}\frac{(1+\alpha^2 l^2)}{l^2}} \\
&= \frac{\alpha}{4G}(r_+ - r_-) \\
&= n_L \hbar \, , \qquad (50) \\
\mathfrak{I}_R &= \int \frac{dM}{\omega_{Rc}} \\
&= \frac{2\alpha}{\lambda}\nu \\
&= \frac{\alpha}{4G}(r_+ + r_-) \\
&= n_R \hbar \, , \qquad (51)
\end{aligned}$$

where n_L, $n_R = 0, 1, 2, ..$. Notice that we have $n_R \geq n_L$.

Firstly, we find that the total horizon area is quantized and equally spaced. The total horizon area is given by

$$A_{tot} \equiv A_{out} + A_{in} = 4\pi\alpha(r_+ + r_-) \, . \tag{52}$$

So, according to the second quantization condition (51), one can obtain

$$A_{tot} = 16\,\pi G\, n_R\,\hbar \, , \qquad n_R = 0, 1, 2, \ldots \tag{53}$$

Next, we find that the difference of the two horizon areas is also quantized and equally spaced due to the first quantization condition. The area difference is given by

$$A_{sub} \equiv A_{out} - A_{in} = 4\,\pi\alpha(r_+ - r_-)\ . \tag{54}$$

Therefore, the area difference is quantized by

$$A_{sub} = 16\,\pi G\, n_L\, \hbar\ \ , \qquad n_L = 0, 1, 2, \ldots \tag{55}$$

From (53) and (55), we further find the quantizations of the outer and inner horizon areas as

$$A_{out} = 8\pi G(n_R + n_L)\,\hbar\ \ , \quad A_{in} = 8\pi G(n_R - n_L)\,\hbar\ , \tag{56}$$

where $n_R \geq n_L$. Therefore we find that both the outer and the inner horizon areas are equally spaced with the same spacing

$$\triangle A_{out} = 8\pi G\hbar\ \ , \quad \triangle A_{in} = 8\pi G\hbar\ . \tag{57}$$

Then, the entropy spectrum of three dimensional Gödel black hole is given by

$$S = \frac{A_{out}}{4G} = 2\,\pi\,(n_R + n_L)\hbar\ . \tag{58}$$

Hence it also has equal spacing, consistent with Bekenstein's proposal

$$\triangle S = 2\,\pi\hbar\ . \tag{59}$$

These results indicate that the spacing of entropy spectrum for three dimensional Gödel black hole is consistent with that for BTZ black hole [21] and warped AdS black hole [22]. However, the quantization process from different viewpoints will lead to different conclusions. For example, when taking the real part of the quasinormal frequency as vibrational frequency for the non-rotating BTZ black hole, the equally spaced mass spectrum and the non-equally spaced area spectrum are obtained in [24].

VI. CONCLUSION

As an exact solution to Einstein-Maxwell-Chern-Simons theory in 2+1 dimensions with a negative cosmological constant, three dimensional Gödel black holes are interesting to study because they display the same peculiar properties as their higher dimensional counterparts.

In this paper, we have studied the perturbation of scalar field and spinor field in the background of three dimensional Gödel black hole. It is shown that the wave equations of scalar field and spinor field can be exactly solved in terms of hypergeometric functions. The quasinormal modes of scalar field and spinor field are analytically calculated by imposing the Dirichlet boundary condition at spatial infinity, which are shown to be of the same form in both cases. By considering the physical interpretation of quasinormal modes, we obtain the consistent transition frequencies from the quasinormal modes of scalar field and spinor field. As an application of quasinormal modes, we use the proposal of area quantization of rotating black hole in [21, 22] to investigate area and entropy quantization of three dimensional Gödel black hole. It has been discussed in Sec.II that there is some amount of ambiguity as to the conserved mass in three dimensional Gödel black hole. However, it is shown that, when the conserved mass of three dimensional Gödel black hole is taken as the parameter ν, the area and entropy can be quantized and the spectra are equally-spaced.

Acknowledgement

The author is grateful to Ming-Fan Li, Shi-Xiong Song, Pu-Jian Mao and Lin-Yu Jia for useful discussions.

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